

# How brokers can optimally plot against traders

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## Abstract

Traders buy and sell financial instruments in hopes of making profit, and brokers are responsible for the transaction. Some brokers, known as market-makers, take the position opposite to the trader's. If the trader buys, they sell; if the trader sells, they buy. Said differently, brokers make money whenever their traders lose money. From this somewhat strange mechanism emerge various conspiracy theories, notably that brokers manipulate prices in order to maximize their traders' losses. In this paper, our goal is to perform this evil task optimally. Assuming total control over the price of an asset (ignoring the usual aspects of finance such as market conditions, external influence or stochasticity), we show how in cubic time, given a set of trades specified by a stop-loss and a take-profit price, a broker can find a maximum loss price movement. We also study the same problem under a model of probabilistic trades. We finally look at the online trade setting, where broker and trader exchange turns, each trying to make a profit. We show that the best option for the trader is to never trade.

## 1 Introduction

Trading is the practice of buying or selling financial assets with the aim of making a profit. A trader can buy an instrument at some price  $p$  and sell it at price  $p'$ , making a profit of  $p' - p$  (which might be negative). The trader can also sell an instrument not even in his possession, say at price  $p$ , with the obligation to buy it back someday, say at a time where the new price is  $p'$ , making a profit of  $p - p'$  (this is called *shorting* in the trading jargon). A *broker* is usually responsible for the execution of a trade, taking care of the technical aspects of the transaction. There are multiple ways of handling these details, but there is a particular type of broker, called *market-makers*, that may do so by placing a trade in the opposite direction of the trader. To put it simply, if a trader wants to buy an asset, the market-maker will sell it, and if the trader wants to sell it, they will buy it. After all, there has to be two parties involved in a transaction, and market-makers assume one of the roles <sup>1</sup>. Now, this puts brokers in a rather peculiar position. By mirroring the trades that they are paid to execute, they win money when their traders lose money, and lose money when their traders win money, giving them an incentive for their clients to perform poorly. Even worse, if they were able to manipulate prices, perhaps they would incur as much losses as possible to their traders! In fact, if one searches the Internet long enough, one can find articles and forum discussions involving conspiracy theorists that believe that brokers, having access to large amounts of capital and trader information, do influence prices in their favor somehow. These ideas are reinforced by statistics showing that a majority of traders lose money <sup>2</sup>.

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<sup>1</sup> This is especially prevalent in markets in which there are no restrictions on buying versus selling, for instance the currency exchange market. Unlike the stocks market where regulations on shorting may apply, buying Euros with US dollars is not more or less restricted than selling Euros to buy US dollars

<sup>2</sup> See e.g. : [https://www.dailyfx.com/forex/fundamental/article/special\\_report/2015/06/25/what-is-the-number-one-mistake-forex-traders-make.html](https://www.dailyfx.com/forex/fundamental/article/special_report/2015/06/25/what-is-the-number-one-mistake-forex-traders-make.html)



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Taking such accusations seriously is likely to make an economist jump out of his seat, since there are too many factors driving asset prices for a single entity to control it. But here, we rather take an algorithmic perspective on these theories. To take it to the extreme, suppose that brokers have total control over the prices. That is, armed with the knowledge of every trade that is currently active, their goal is to make people lose as much money as possible. The question is: even with the ultimate power of price manipulation, *can they*? Is this optimization problem easy? In this paper, we wear the hat of (would-be) mischievous brokers and devise price manipulation algorithms to do our evil bidding.

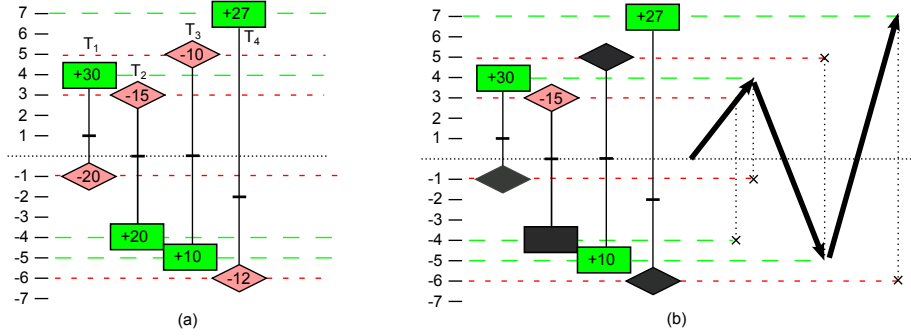
More precisely, we model trades as bounded by two closing prices - a winning and a losing price. This is typical in trading: traders want to limit their risk and often set up a *stop-loss*, a price at which the trade closes automatically when too much losses are incurred. This is usually accompanied with a *take-profit* price, which closes the trade when its profit is high enough. Brokers can use these two pieces of information to their advantage, leading to three types of problems that we study here. First, given a set of bounded trades, what is the price movement that maximizes trader losses? We show in Section 2 that this problem reduces to finding a maximum independent set in a bipartite graph, yielding a  $O(n^3)$  time algorithm [4, 5], where  $n$  is the number of trades. This question has another interpretation: as a trader, what is the worst that could happen with a set of opened trades on a single asset? There seems to be no answer in the literature for this simple question. Algorithms exist for the seemingly related notions of *value at risk* [2] and *maximum loss* [6], but these measures are based on stochastic prices, operate on multiple assets and are subject to various market conditions, unlike here. In Section 3, we make the trades probabilistic. That is, each price has a given probability of being a winning or a losing price, and the goal is to maximize the expected losses incurred on traders. Interestingly, many papers model prices as stochastic (e.g. as a Brownian motion [1]), but never the trades. We devise a  $O(n^3 S^3)$  algorithm again based on bipartite graphs, where  $S$  is the size of the support of the trades, and show that if winning and losing prices are uniformly distributed, this task can be accomplished in time  $O(n^6)$ . Finally in Section 4, we look at the *online* setting. That is, the trader can add a new trade at any given time, and the broker still has to maximize losses. This leads to a two-player game where the trader and broker exchange turns, and we conclude that the optimal play for the trader is to never trade - the broker never loses.

## 2 The Maximum Trader Loss Problem

In trading, prices are usually handled to the fourth or fifth decimal, and as they are not continuous we will consider them as integers, the current price being 0. In this paper, all trades are done on a single asset, and no fee is required to open a trade. Although this work is motivated by loss maximization, we will speak of maximizing profit - both because losses are actually the broker's profit, and because there is some form of cognitive dissonance involved for a computer scientist in "maximizing losses".

A trade  $T = (o_T, w_T, \ell_T, s_T)$  consists of four parameters:  $o_T$  is the opening price,  $w_T$  is the winning price,  $\ell_T$  is the losing price and  $s_T$  is the size of the trade (sometimes called the *volume*), which weights the profit or loss made from the trade. We require that either  $w_T > o_T > \ell_T$  if  $T$  is a *buy*, or  $w_T < o_T < \ell_T$  if  $T$  is a *sell*. Moreover, none of  $w_T$  or  $\ell_T$  can be between  $o_T$  and the current price 0. We denote  $\text{sgn}_T = 1$  if  $w_T > o_T > \ell_T$  and  $\text{sgn}_T = -1$  if  $w_T < o_T < \ell_T$ . A trade session consists of a set of trades  $\mathcal{T}$  and a price movement  $M$ , which can be described by a sequence of ups and downs  $(m_1, \dots, m_k)$ , where  $m_i \in \{+1, -1\}$  for all  $1 \leq i \leq k$ . The current price after the  $j$ -th movement is  $\sum_{i=1}^j m_i$ .

Given a price movement  $M$ , a trade  $T$  closes either at price  $p = w_T$  or  $p = \ell_T$ , whichever is reached first by  $M$ . The closing price  $p \in \{w_T, \ell_T\}$  implied by  $M$  is denoted  $cl_T(M)$ . The broker's profit made from  $T$  in a given price movement  $M$  is  $pr_T(M) = \text{sgn}_T(cl_T(M) - o_T) s_T$ . We say that  $T$  is *won* when  $pr_T(M)$  is positive and *lost* otherwise. This is the “real-life” profit calculation we are interested in, but some of our algorithms work on arbitrary profit functions “for free”. We will use  $\tilde{pr}_T(M)$  to denote any function of  $T$  and  $M$  that can take two values  $\tilde{pr}_T^+$  or  $\tilde{pr}_T^-$ , which must satisfy  $\tilde{pr}_T(M) = \tilde{pr}_T^+ > 0$  if  $cl_T(M) = w_T$  and  $\tilde{pr}_T(M) = \tilde{pr}_T^- < 0$  if  $cl_T(M) = \ell_T$  (in words, wins are positive and losses are negative). Note that  $pr_T(M)$  is such a function.



■ **Figure 1** (a) An example input for the MTL problem with  $|\mathcal{T}| = 4$ . The display order of the trades is immaterial - they are all received simultaneously. The opening prices are indicated by a dent in the middle of the trades. Each trade has a winning price (long green dashes) and a losing price (short red dashes). The profit made by winning a trade is indicated in the green box, and the profit made by losing it in the red diamond. For instance,  $T_1$  has  $o_{T_1} = 1$ ,  $w_{T_1} = 4$ ,  $\ell_{T_1} = -1$  and  $s_{T_1} = 10$  (sizes are not shown, but are implicitly defined by the profit/loss values). (b) An optimal price movement  $M$  for  $\mathcal{T}$  that makes a total profit of  $30 - 15 + 10 + 27 = 52$ . The green/red lines now indicate the lifespan of each trade according to  $M$ . The profit values not realized by  $M$  are grayed-out. Only  $T_2$  is lost since price 3 is reached before  $-4$ .

A trade session can be seen as a one-turn two player game where the trader first opens a set  $\mathcal{T}$  of trades, and afterwards the broker determines a price movement  $M$ . This price movement occurs and the trades get closed according to it. We require that  $M$  closes every trade eventually. The problem statement follows:

#### Maximum Trader Loss Problem (MTL) :

**Given:** a set of trades  $\mathcal{T}$ ;

**Find:** a price movement  $M$  that maximizes the broker's profit  $\sum_{T \in \mathcal{T}} pr_T(M)$ .

We call a solution  $M$  to MTL an *optimal price movement* for  $\mathcal{T}$ . Some observations about  $M$  are in order here. Consider the trade  $T = (0, 2^n, -2^n, 1)$ , where  $n$  is the size of the input. As  $M$  has to close  $T$ , representing  $M$  as a sequence of unit movements might yield an exponential-size output, which is not necessary. Instead  $M$  can be described as a sequence of prices that it reaches, from which the  $+1/-1$  sequence can easily be inferred. That is, if  $M = (p_1, p_2, \dots, p_k)$ , then  $p_1 = 0$  is the initial price,  $p_2$  is the second price reached after a series of  $|p_2 - p_1|$  unit movements ( $+1$  or  $-1$  depending on whether  $p_2 > p_1$  or  $p_2 < p_1$ ), and so on. We implicitly assume that  $M$  is output in this form - it is easy to verify that the algorithms presented here can do so. As a note, it is not hard to verify that  $M$ , if minimal, makes larger and larger zig-zag movements, and can be described by  $(p_1, \dots, p_k)$  with  $p_i$  and  $p_{i-1}$  having different signs and  $|p_i| > |p_{i-2}|$  for all  $2 < i \leq k$ .

We show here a simple strategy to determine  $M$ : find a set of *compatible* trades to win, and accept a loss on the other trades. We say that a set of trades  $\mathcal{T}'$  is *compatible* if

there exists a price movement  $M$  such that  $pr_T(M) > 0$  for all  $T \in \mathcal{T}'$ . Otherwise  $\mathcal{T}$  is *incompatible*. Pairs of incompatible trades have a very simple, yet useful characterization:

► **Lemma 1.** *Let  $T_1$  and  $T_2$  be two trades. Then  $T_1$  and  $T_2$  are incompatible if and only if  $sgn_{T_1} \neq sgn_{T_2}$ ,  $|\ell_{T_1}| \leq |w_{T_2}|$  and  $|\ell_{T_2}| \leq |w_{T_1}|$ .*

**Proof.** We exhaust every possible case. Suppose first that  $s = sgn_{T_1} = sgn_{T_2}$ . If  $s = 1$  (resp.  $-1$ ), then the price movement that always goes up (resp. down) wins both trades, and they are compatible. So assume instead that  $sgn_{T_1} \neq sgn_{T_2}$ . If  $|\ell_{T_1}| > |w_{T_2}|$ , then the price movement that goes to  $w_{T_2}$  first wins  $T_2$  and does not reach  $\ell_{T_1}$ . It then suffices to go from  $w_{T_2}$  to  $w_{T_1}$ . The same idea applies when  $|\ell_{T_2}| > |w_{T_1}|$ . So finally assume that  $|\ell_{T_1}| \leq |w_{T_2}|$  and  $|\ell_{T_2}| \leq |w_{T_1}|$ . As both trades are of opposite signs, the first price to be hit has to be a loss, making them incompatible. ◀

If a set of trades  $\mathcal{T}'$  has two trades  $T_1, T_2$  that are incompatible, then they cannot both be won and trivially  $\mathcal{T}'$  is not compatible. The converse also holds.

► **Lemma 2.** *Let  $\mathcal{T}'$  be a set of trades such that every pair of trades from  $\mathcal{T}'$  is compatible. Then  $\mathcal{T}'$  is compatible. Moreover, one can construct in time  $O(|\mathcal{T}'|^2)$  a price movement  $M$  such that  $pr_T(M) > 0$  for all  $T \in \mathcal{T}'$ .*

**Proof.** We use induction on  $|\mathcal{T}'|$ . The case  $|\mathcal{T}'| = 1$  is trivial: it suffices to go to the single winning price. Suppose by induction that any proper subset of  $\mathcal{T}'$  is compatible. Denote by  $\mathcal{T}'(p)$  the set of trades of  $\mathcal{T}'$  that close at price  $p$ , winning or losing. Let  $p^+$  be the minimum value above 0 such that  $\mathcal{T}'(p^+) \neq \emptyset$ , and let  $p^-$  be the maximum value below 0 such that  $\mathcal{T}'(p^-) \neq \emptyset$ . If all trades of  $\mathcal{T}'(p^+)$  are won at price  $p^+$ , we can raise the price from 0 to  $p^+$ , win these trades, bring the price back to 0 and apply induction on  $\mathcal{T}' \setminus \mathcal{T}'(p^+)$ . The same applies if all trades of  $\mathcal{T}'(p^-)$  are won at  $p^-$ . So suppose that there are  $T_1 \in \mathcal{T}'(p^+)$  and  $T_2 \in \mathcal{T}'(p^-)$  that are losing. Then  $T_1$  is a sell and  $T_2$  is a buy, and hence  $sgn_{T_1} \neq sgn_{T_2}$ . Moreover,  $w_{T_2} \geq \ell_{T_1} > 0$  by our choice of  $p^+$ , and similarly  $0 > \ell_{T_2} \geq w_{T_1}$ . By Lemma 1,  $T_1$  and  $T_2$  are incompatible, a contradiction.

It is easy to derive a (naive)  $O(|\mathcal{T}'|^2)$  time algorithm from this argument: find  $p^+$  and  $p^-$  by scanning all trades in time  $O(|\mathcal{T}'|)$ , and one of  $\mathcal{T}'(p^+)$  or  $\mathcal{T}'(p^-)$  will contain only winning trades. We can then move the price to the winning position, close the appropriate trades and repeat  $O(|\mathcal{T}'|)$  times. ◀

Lemma 2 essentially states that we can devote our efforts to finding a set of pairwise-compatible trades  $\mathcal{T}'$  that maximizes profit, minus the losses incurred by the trades not in  $\mathcal{T}'$ . We call  $\mathcal{T}'$  an *optimal set of winning trades*. The advantage of this approach is that we don't have to worry about the temporal aspects of price movements, as they will be taken care of by the algorithm provided by Lemma 2.

The *incompatibility graph*  $G = G(\mathcal{T})$  of a set of trades  $\mathcal{T}$  is the graph with vertex set  $\mathcal{T}$ , and an edge between  $T_1$  and  $T_2$  iff they are incompatible. Note that by Lemma 1, testing compatibility on a pair of trades can be done in constant time, and thus  $G$  can be built in time  $O(|\mathcal{T}|^2)$ . Now, as per Lemma 2,  $\mathcal{T}' \subseteq \mathcal{T}$  is compatible if and only if  $\mathcal{T}'$  forms an independent set in  $G$  (a set of vertices with no shared edges). Finding an independent set of maximum weight is NP-Hard, but we are saved by the following observation:

► **Lemma 3.** *Let  $\mathcal{T}$  be a set of trades. Then  $G(\mathcal{T})$  is bipartite.*

**Proof.** Let  $\mathcal{T}^+ = \{T \in \mathcal{T} : sgn_T = 1\}$  and  $\mathcal{T}^- = \mathcal{T} \setminus \mathcal{T}^+$ . Since any pair of trades from  $\mathcal{T}^+$  are of the same sign, they are pairwise-compatible by Lemma 1. The same holds for any pair of trades from  $\mathcal{T}^-$ . Therefore,  $\mathcal{T}^+$  and  $\mathcal{T}^-$  form a bipartition of  $V(G(\mathcal{T}))$ . ◀

There are known polynomial-time algorithms that find a maximum weight independent set  $I$  in a bipartite graph. But before we rush in the reduction that seems natural, in our setting each trade that is *not* won is lost and adds a negative profit  $q$ , i.e. each vertex not chosen to be in  $I$  must be penalized by a weight of  $q$ . This is a bit different than solely finding a maximum weight independent set, and it is not immediately clear how this problem can be handled using known algorithms. We show that it suffices to adjust the weights accordingly.

► **Theorem 4.** *Let  $\mathcal{T}$  be a set of trades and  $G = G(\mathcal{T})$  with each trade  $T \in \mathcal{T}$  having a profit function  $\tilde{p}r_T(M)$ . Let  $f : \mathcal{T} \mapsto \mathbb{R}$  be a weighting function that assigns weight  $f(T) = \tilde{p}r_T^+ - \tilde{p}r_T^-$  to every trade  $T \in \mathcal{T}$ . Then  $\mathcal{T}'$  is an optimal set of winning trades if and only if  $\mathcal{T}'$  is a maximum weight independent set in  $G$  with respect to  $f$ .*

**Proof.** Let  $M$  be a price movement winning every trade of  $\mathcal{T}'$ , and let  $\overline{\mathcal{T}'} = \mathcal{T} \setminus \mathcal{T}'$  be the set of lost trades. Since  $\mathcal{T}'$  is optimal, it maximizes

$$\begin{aligned} \sum_{T \in \mathcal{T}} \tilde{p}r_T(M) &= \sum_{T \in \mathcal{T}'} \tilde{p}r_T^+ + \sum_{T \in \overline{\mathcal{T}'}} \tilde{p}r_T^- = \sum_{T \in \mathcal{T}'} \tilde{p}r_T^+ + \left( \sum_{T \in \mathcal{T}} \tilde{p}r_T^- - \sum_{T \in \mathcal{T}'} \tilde{p}r_T^- \right) \\ &= \sum_{T \in \mathcal{T}'} (\tilde{p}r_T^+ - \tilde{p}r_T^-) + \sum_{T \in \mathcal{T}} \tilde{p}r_T^- = \sum_{T \in \mathcal{T}'} f(T) + \sum_{T \in \mathcal{T}} \tilde{p}r_T^- \end{aligned}$$

among all possible choices of  $\mathcal{T}'$ . Now, because  $\sum_{T \in \mathcal{T}} \tilde{p}r_T^-$  does not depend on the choice of  $\mathcal{T}'$ , it can be considered as a constant and thus  $\mathcal{T}'$  is optimal if and only if it maximizes  $\sum_{T \in \mathcal{T}'} f(T)$ , i.e. if and only if it is a maximum weight independent set in  $G(\mathcal{T})$ . ◀

Now, the traditional way of finding a maximum weight independent set in a bipartite graph  $G$  is done by finding a minimum-cut/maximum-flow on a slightly modified version of  $G^3$ . This can be implemented to run in time  $O(|T|^3)$  using the Stoer-Wagner min-cut algorithm [5] or a variety of max-flow algorithms, e.g. [4].

### 3 Probabilistic trades

One may argue that traders do not open their trades and simply wait for the price to reach the winning or losing price. Traders are sometimes emotive and may cut their profit or loss ahead of time, or leave them open a bit longer ‘just in case’. For instance, a movement in the losing direction may induce the fear that more losses are to come, hence triggering an early trade close. Or, when the price is about to reach the losing trigger, the trader might get confident that the price is about to turn around, pushing the losing price further.

This idea adds uncertainty in the determination of the optimal price movement. To model this situation, we will assume that each trade is accompanied with a probability distribution on the winning and closing prices. These could, for example, be computed by the broker by analysis the past behavior of each trader. More precisely, a *probabilistic trade* is given by  $T = (o_T, f_T, g_T, s_T, \text{sgn}_T)$  where  $o_T$  and  $s_T$  are the opening price and the size as before,  $f_T(p) = \Pr[w_T = p]$  is a probability mass function representing the probability that  $T$  closes with a win when the price reaches  $p$ ,  $g_T(q) = \Pr[\ell_T = q]$  is a probability mass function representing the probability that  $T$  closes with a loss when the price reaches  $q$ , and

<sup>3</sup> This method, however, seems too “trivial” to be published anywhere, and hence there is no paper to cite. To give credit where it is due, the ideas were taken from <http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-854j-advanced-algorithms-fall-2008/assignments/sol5.pdf>

$\text{sgn}_T \in \{1, -1\}$  is made explicit. If  $T$  is a buy, we require that  $f_T(p) = 0$  for  $p \leq o_T$  and  $g_T(q) = 0$  for  $q \geq o_T$ , and if  $T$  is a sell we require  $f_T(p) = 0$  for  $p \geq o_T$  and  $g_T(q) = 0$  for  $q \leq o_T$ . Moreover  $f_T(q) = g_T(q) = 0$  for  $q$  between  $o_T$  and 0. We assume that  $f_T$  and  $g_T$  are independent. Note that probabilistic trades are a generalization of what we will now call the *deterministic trades* described in the previous section.

A trade session can now be seen as such: (1) the broker receives a set of probabilistic trades  $\mathcal{T}$  and decides on a price movement  $M$ ; (2) the trader randomly picks, independently, a winning and losing price for each trade  $T$  according to  $f_T$  and  $g_T$ , respectively; (3) the price movement  $M$  occurs and each trade closes at the prices determined at step 2.

In this setting, a natural option for the broker is to maximize his expected gain. For a probabilistic trade  $T$ , denote by  $T(w, \ell)$  the deterministic trade obtained by fixing  $w_T = w$  and  $\ell_T = \ell$ . For a given price movement  $M$  and a trade  $T$ , the expected profit of  $T$  is  $\mathbb{E}[pr_T(M)] = \sum_{w, \ell} f_T(w)g_T(\ell)pr_{T(w, \ell)}(M)$  where  $w$  and  $\ell$  range over all possible prices. The problem now becomes the following:

**Maximum Expected Trader Loss Problem (METL) :**

**Given:** a set of probabilistic trades  $\mathcal{T}$ ;

**Find:** a price movement  $M$  maximizing the expected profit  $\mathbb{E}[pr_{\mathcal{T}}(M)] = \mathbb{E}[\sum_{T \in \mathcal{T}} pr_T(M)]$ .

For a trade  $T$ , denote by  $\mathcal{S}(T)$  the *support* of  $T$ , that is the set of prices  $p$  such that one of  $f_T(p) > 0$  or  $g_T(p) > 0$  holds. If  $\mathcal{T}$  is a set of trades, let  $\mathcal{S}(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} \mathcal{S}(T)$ . We may write  $\mathcal{S}$  instead of  $\mathcal{S}(\mathcal{T})$ . We assume that  $\mathcal{S}$  is finite. We show that if  $\mathcal{S}$  is not too large, then probabilistic trades can also be handled by reducing the problem to independent sets in bipartite graphs. The idea is to simply decompose each trade  $T$  into its set of possible deterministic trades, and solve the underlying deterministic trade setup as before. Note that this is only possible under the assumption that prices (and hence  $f_T$  and  $g_T$ ) are discrete and  $\mathcal{S}$  is finite. Other cases remain open.

For  $T \in \mathcal{T}$  and prices  $w$  and  $\ell$ , denote by  $T^*(w, \ell) = (o_T, w, \ell, f_T(w)g_T(\ell)s_T)$  the deterministic trade corresponding to  $T$  with  $w_T = w, \ell_T = \ell$  and with size weighted by the chances of  $w$  and  $\ell$  occurring. Let  $T^* = \bigcup_{w, \ell \in \mathcal{S}} T^*(w, \ell)$  and  $\mathcal{T}^* = \bigcup_{T \in \mathcal{T}} T^*$ .

► **Lemma 5.** *A price movement  $M$  is optimal for a set of probabilistic trades  $\mathcal{T}$  if and only if  $M$  is optimal for the set of deterministic trades  $\mathcal{T}^*$ .*

**Proof.** Observe first that for some trade  $T \in \mathcal{T}$ , prices  $w$  and  $\ell$  and price movement  $M$ ,  $pr_{T(w, \ell)}(M) = \frac{1}{f_T(w)g_T(\ell)}pr_{T^*(w, \ell)}(M)$ , because of the size adjustment. Now, starting with the maximization objective for  $\mathcal{T}$ , using linearity of expectation we get

$$\mathbb{E} \left[ \sum_{T \in \mathcal{T}} pr_T(M) \right] = \sum_{T \in \mathcal{T}} \sum_{w, \ell} f_T(w)g_T(\ell)pr_{T(w, \ell)}(M) = \sum_{T \in \mathcal{T}} \sum_{w, \ell} pr_{T^*(w, \ell)}(M) = \sum_{T \in \mathcal{T}^*} pr_T(M)$$

which is exactly the maximization objective for  $\mathcal{T}^*$ . Hence both objectives are equal. ◀

Using the results from section 2, the METL problem can be solved in time  $O(|\mathcal{T}^*|^3) = O((|\mathcal{T}||\mathcal{S}|)^3)$ . One might then (legitimately) ask: is it good? This question deserves a bit of digressing. If  $\mathcal{S}$  is large, then this is not really good. But one might argue that  $\mathcal{S}$  has to be given in the input in the form of two distributions for every trade. It is then reasonable to expect our algorithm to depend on  $|\mathcal{S}|$  if, say,  $f_T$  and  $g_T$  are given as two lists of price-probability pairs for each trade - then the running time is polynomial in the size of the input. But what if  $f_T$  and  $g_T$  can be expressed in a compact form? Maybe  $f_T$  (or  $g_T$ ) could be given as the code of a function that computes  $f_T(p)$  for any price  $p$ , which takes constant space. For instance, the distribution  $f_T(p) = 1/2^{|\mathcal{T}|}$  for all prices  $0 < p \leq 2^{|\mathcal{T}|}$



is easy to describe in a few bits, but makes  $|\mathcal{S}|$  exponential. This discussion leads to the following question: is there an algorithm for METL that is *always* polynomial in the input size? We do not have a general answer, but in the next section, we show that it is the case for trades opening at 0 and having uniformly distributed closing prices, which are perhaps the simplest (yet surprisingly difficult to handle) probabilistic trades yielding exponential  $|\mathcal{S}|$  from a short description.

### Uniform trades

A *uniform trade*  $T = (\hat{w}_T, \hat{\ell}_T, s_T)$  is a probabilistic trade with opening price  $o_T = 0$ , size  $s_T$  and  $\text{sgn}_T = \frac{\hat{w}_T}{|\hat{w}_T|}$ , and where  $f_T$  (resp.  $g_T$ ) is, implicitly, the uniform distribution across values between 0 and  $\hat{w}_T$  (resp. 0 and  $\hat{\ell}_T$ ). That is,  $f_T(p) = 1/|\hat{w}_T|$  for all  $p$  between 0 and  $\hat{w}_T$  (excluding 0), and  $g_T(p) = 1/|\hat{\ell}_T|$  for all  $p$  between 0 and  $\hat{\ell}_T$  (excluding 0). Note that  $T^*$  has a trade of size  $\frac{s_T}{|\hat{w}_T||\hat{\ell}_T|}$  for each price pair  $p, q$  with  $0 < p \leq \max(\hat{w}_T, \hat{\ell}_T)$  and  $0 > q \geq \min(\hat{w}_T, \hat{\ell}_T)$ . In the rest of this section,  $\mathcal{T}$  denotes a set of uniform trades. We show that even though  $\mathcal{S}$  might have exponential size,  $\mathcal{T}$  can be handled in time  $O(|\mathcal{T}|^6)$ .

Denote  $\hat{\mathcal{S}} = \bigcup_{T \in \mathcal{T}} \{\hat{w}_T, \hat{\ell}_T\}$ . The idea is to “collapse” every trade of  $\mathcal{T}^*$  to some price of  $\hat{\mathcal{S}}$ , which is desirable since there are  $O(|\mathcal{T}|)$  of them (at most two prices per trade). Lemma 6 shows that this is allowable. The proof is somewhat involved, in contrast with the other ones in this paper, and we leave it to the Appendix.

► **Lemma 6.** *If  $\mathcal{T}$  is a set of uniform trades, then there is an optimal price movement  $M$  for  $\mathcal{T}^*$  such that  $M$  only changes direction at prices that are in  $\hat{\mathcal{S}}$ .*

Let  $\hat{\mathcal{S}}^+ = (p_1, \dots, p_n)$  (resp.  $\hat{\mathcal{S}}^- = (q_1, \dots, q_m)$ ) be the prices of  $\hat{\mathcal{S}}$  that are above 0, sorted in increasing order (resp. below 0, sorted in decreasing order). Fix  $p_0 = q_0 = 0$ . Now, let  $M$  be an optimal price movement for  $\mathcal{T}^*$  as described in Lemma 6. Then  $M$  alternates between prices in  $\hat{\mathcal{S}}^+$  and  $\hat{\mathcal{S}}^-$ . Let  $T \in \mathcal{T}$  be a buy trade, and let  $T' = T^*(p_{i-1}+k, q_{j-1}-h) \in \mathcal{T}^*$  be a corresponding deterministic trade with  $w_{T'} = p_{i-1}+k < p_i$  and  $\ell_{T'} = q_{j-1}-h > q_j$  (assume  $k, h > 0$ ). One implication of Lemma 6 is that since the closing prices of  $T'$  are lying between two prices of  $\hat{\mathcal{S}}$ ,  $T'$  is won (resp. lost) by  $M$  if and only if  $T^*(w, \ell)$  is won (resp. lost) by  $M$  for every  $p_{i-1} < w \leq p_i$  and every  $q_{j-1} > \ell \geq q_j$ . The same idea applies to sell trades, and thus all such trades can be collapsed into one using the following scheme.

For a trade  $T \in \mathcal{T}$  and prices  $p_i \in \hat{\mathcal{S}}^+ \cap \mathcal{S}(T)$ ,  $q_j \in \hat{\mathcal{S}}^- \cap \mathcal{S}(T)$  with  $i, j > 0$ , let  $P_T = \sum_{k=p_{i-1}+1}^{p_i} \sum_{h=q_j}^{q_{j-1}-1} s_{T^*(k,h)} k = \frac{(q_{j-1}-q_j)s_T}{2|\hat{w}_T||\hat{\ell}_T|} (p_i^2 + p_i - p_{i-1}^2 - p_{i-1})$  and  $Q_T = \sum_{k=q_j}^{q_{j-1}-1} \sum_{h=p_{i-1}+1}^{p_i} s_{T^*(k,h)} k = \frac{(p_i-q_{i-1})s_T}{2|\hat{w}_T||\hat{\ell}_T|} (-q_j^2 + q_j + q_{j-1}^2 - q_{j-1})$ . Denote by  $\hat{T}(p_i, q_j) = (0, p_i, q_j, s_T)$  the deterministic trade accompanied by the profit function  $\tilde{p}r_T(M)$  such that if  $T$  is a buy,  $\tilde{p}r_T^+ = P_T$  and  $\tilde{p}r_T^- = Q_T$ , and if  $T$  is a sell,  $\tilde{p}r_T^+ = -Q_T$  and  $\tilde{p}r_T^- = -P_T$ .

One can verify that winning (resp. losing)  $\hat{T}(p_i, q_j)$  corresponds to winning (resp. losing) every trade of  $\mathcal{T}^*$  having one closing price in the interval  $(p_{i-1}, p_i]$  the other in  $[q_j, q_{j-1})$ , the important point being that these values can be computed quickly. This leads to the next Theorem, which follows from the discussion above. We omit the full details.

► **Theorem 7.** *Let  $M$  be a price movement that is optimal for  $\hat{\mathcal{T}} = \bigcup_{(p_i, q_j) \in \hat{\mathcal{S}}^+ \times \hat{\mathcal{S}}^-} \hat{T}(p_i, q_j)$  and, for each  $T \in \hat{\mathcal{T}}$ , the corresponding profit function  $\tilde{p}r_T(M)$  as described above. Then  $M$  is optimal for  $\mathcal{T}^*$ , and hence also for  $\mathcal{T}$ .*

As  $\hat{\mathcal{S}}$  has  $O(|\mathcal{T}|)$  values, one can check that  $\hat{\mathcal{T}}$  and profit functions can be constructed in time  $O(|\mathcal{T}|^6)$  (a generous upper bound) and has  $O(|\mathcal{T}|^2)$  trades. Solving MTL on  $\hat{\mathcal{T}}$  then takes time  $O(|\mathcal{T}|^6)$ , which, despite not looking too attractive, is independent of  $|\mathcal{S}|$ .

#### 4 Online trades

In this section, all trades are deterministic. We now allow the trader to interrupt the broker's price manipulation at any time in order to add a trade. Under this model, it is not hard to devise examples in which the previous algorithms may fail, i.e. make the broker lose money. Here we are only concerned with who makes profit - not necessarily the optimality. The question is: given his new power, can the trader make money? There is an "obvious" answer: if there was a strategy for the trader that guarantees profit even against prices conspiring against him, someone would have figured it out by now. But surprisingly, there doesn't seem to be a clear answer in the literature. One impossibility result is given by the *efficient market hypothesis* (EMH) [3], which essentially states that if prices perfectly reflect their environment, then no strategy can win *consistently*, i.e. against every price movement. In the same vein, assuming the price is a random walk, the best trader strategy has expected profit 0, so there exists a price movement with no win for the trader. Thus all the broker has to do is "mimic" a worst-case EMH-driven price, or random walk. How to do this is not clear though. Moreover, if the trader is allowed to open an infinity of trades, it can be shown that a Martingale betting system has expected profit strictly above 0 against a random walk. Here we make no assumption on the finiteness of the trades (though the trader loses in an infinite game) and provide an explicit strategy for the broker.

This problem can be seen as a game described below. For a given price  $p$ , the value of a trade  $T$  at price  $p$  is  $val(T, p) = sgn_T(p - o_T)s_T$ , and represents the profit the broker would get if  $T$  was closed at price  $p$ . The price starts at 0 and the trade set  $\mathcal{T}$  is initially empty. A *turn* starts with the trader adding trades to  $\mathcal{T}$  (or possibly none), and the broker, moving second, decides if the price goes either up or down by 1. If the price  $p$  after this move is equal to either  $w_T$  or  $\ell_T$  for an open trade  $T$ , then  $val(T, p)$  is added to the broker's gains  $g$  and  $T$  is removed from  $\mathcal{T}$ . The game ends when a turn finishes with no opened trades. A *position*  $\mathcal{P} = (\mathcal{T}, g, p)$  is described by a set of open trades  $\mathcal{T}$ , the total gain  $g$  currently accumulated by the broker, and the current price  $p$ . The value of  $\mathcal{P}$  is  $val(\mathcal{P}) = \sum_{T \in \mathcal{T}} val(T, p)$  and its *total value* is  $Val(\mathcal{P}) = val(\mathcal{P}) + g$ . The  $i$ -th turn is described by 3 positions  $\mathcal{P}_i = (\mathcal{T}_i, g_i, p_i)$ ,  $\mathcal{P}'_i = (\mathcal{T}'_i, g'_i, p'_i)$  and  $\mathcal{P}''_i = (\mathcal{T}''_i, g''_i, p''_i)$  respectively corresponding to the positions before that trader's turn, after his turn, and after the broker's turn. Note that  $\mathcal{P}_{i+1} = \mathcal{P}''_i$ . The trader wins the game if and only if the game ends at turn  $j$  with  $g''_j < 0$ . Thus if the game never ends (which we do not rule out), then the trader does not win.

We first address the question of losing and winning prices. One could argue that a trader might have a better strategy if he was able to close an opened trade  $T$  at any time, instead of having to wait for the price to reach  $w_T$  or  $\ell_T$ . The next results shows that this is not the case, since the trader can always *simulate* closing a trade at the current price.

► **Lemma 8.** *Let  $p$  be the current price. For any trade  $T$ , the trader can add a trade  $T'$  such that for any price movement  $M$ ,  $pr_T(M) + pr_{T'}(M) = val(T, p)$ .*

**Proof.** The desired trade is simply  $T' = (p, \ell_T, w_T, s_T)$ . Note that  $sgn_T = -sgn_{T'}$ , and that both trades will have the same closing price  $p' = cl_T(M) = cl_{T'}(M)$  whatever the price movement  $M$  is. Thus  $pr_T(M) + pr_{T'}(M) = sgn_T(p' - o_T)s_T - sgn_T(p' - p)s_T = sgn_T(p - o_T)s_T = val(T, p)$ . ◀

This lets us characterize winning and losing games quite easily:

► **Lemma 9.** *The trader can win if and only if there is a turn  $i$  that ends with  $Val(\mathcal{P}''_i) < 0$ .*



**Proof.** ( $\Leftarrow$ ) If every turn ends in a position  $\mathcal{P}_i''$  with  $\text{Val}(\mathcal{P}_i'') \geq 0$ , then either the game never ends or, in particular, the very last position  $\mathcal{P}_j''$  ends with no open trade and hence  $\text{val}(\mathcal{P}_j'') = 0$ . This implies that  $0 \leq \text{Val}(\mathcal{P}_j'') = \text{val}(\mathcal{P}_j'') + g_j'' = g_j''$  and thus the broker won.

( $\Rightarrow$ ) Suppose that  $\text{Val}(\mathcal{P}_i'') = \text{val}(\mathcal{P}_i'') + g_i'' < 0$  for some  $i$ . If the trader was able to close all of his trades at their current price, then he would win. This can easily be accomplished by applying Lemma 8 on every trade of  $\mathcal{T}_{i+1}$ .  $\blacktriangleleft$

Lemma 9 clarifies the goal of the broker: prevent, at all costs, the game to reach a trade setup with a losing total value. We show that the broker always has a good option.

► **Lemma 10.** *Let  $\mathcal{P}_i$  be the position when turn  $i$  starts. Denote by  $\mathcal{P}_{i+1}^+$  the  $(i+1)$ -th position reached if the broker raises the price, and by  $\mathcal{P}_{i+1}^-$  the  $(i+1)$ -th position reached if the broker lowers the price. Then  $\text{Val}(\mathcal{P}_{i+1}^+) - \text{Val}(\mathcal{P}_i) = -(\text{Val}(\mathcal{P}_{i+1}^-) - \text{Val}(\mathcal{P}_i))$ .*

*In particular, at least one of  $\text{Val}(\mathcal{P}_{i+1}^+) \geq \text{Val}(\mathcal{P}_i)$  or  $\text{Val}(\mathcal{P}_{i+1}^-) \geq \text{Val}(\mathcal{P}_i)$  holds.*

**Proof.** Let  $d \in \{1, -1\}$ , and let  $\mathcal{P}_{i+1}^d = \mathcal{P}_{i+1}^+$  if  $d = 1$ , and  $\mathcal{P}_{i+1}^d = \mathcal{P}_{i+1}^-$  if  $d = -1$ . The price at the start of turn  $i+1$  is  $p+d$ . Let  $\mathcal{P}_i'$  be the position reached after the trader's turn, and note that since every trade in  $\mathcal{T}_i' \setminus \mathcal{T}_i$  has  $\text{val}(T, p) = 0$ , we have  $\text{Val}(\mathcal{P}_i') = \text{Val}(\mathcal{P}_i)$ . Let  $\mathcal{T}_O^d$  be the trades open in both  $\mathcal{P}_i'$  and  $\mathcal{P}_{i+1}^d$ , and  $\mathcal{T}_C^d$  be the trades open in  $\mathcal{P}_i'$  but closed in  $\mathcal{P}_{i+1}^d$ . Note that  $\mathcal{T}_i' = \mathcal{T}_O^1 \cup \mathcal{T}_C^1 = \mathcal{T}_O^{-1} \cup \mathcal{T}_C^{-1}$ . We show that  $\text{val}(\mathcal{P}_{i+1}^d) - \text{val}(\mathcal{P}_i') = \sum_{T \in \mathcal{T}_O^d \cup \mathcal{T}_C^d} d \cdot s_T$ , which proves the Lemma. If  $T \in \mathcal{T}_O^d$ , then the value of  $T$  varies by  $\text{val}(T, p+d) - \text{val}(T, p) = ds_T$  from  $\mathcal{P}_i'$  to  $\mathcal{P}_{i+1}^d$ . If  $T \in \mathcal{T}_C^d$ , then  $T$  is closed, meaning that  $\text{val}(T, p+d)$  is added to the broker's gains  $g_{i+1}$ , while  $\text{val}(T, p)$  is subtracted from  $\text{val}(\mathcal{P}_{i+1}^d)$ . The difference implied by  $T$  from  $\mathcal{P}_i'$  to  $\mathcal{P}_{i+1}^d$  is again  $ds_T$ .  $\blacktriangleleft$

Since the game starts empty,  $\text{Val}(\mathcal{P}_1) = 0$ , and by Lemma 10, the broker always has an option for this value to never decrease: it suffices, at each turn  $i$ , to find which of  $\mathcal{P}_{i+1}^+$  or  $\mathcal{P}_{i+1}^-$  has a better total value. Then with Lemmas 9 we get the following.

► **Theorem 11.** *The broker cannot lose if he plays optimally.*

## 5 Conclusion

We have provided some useful algorithmic tools for malevolent brokers to plot against traders, but there are still many open questions and directions. For starters, can the  $O(|\mathcal{T}|^3)$  complexity be improved for MLT? As for METL, what is the relationship between the input size,  $|\mathcal{S}|$  and the time required to construct an optimal price movement? Is there a set of probabilistic trades  $\mathcal{T}$  that take space polynomial in  $\mathcal{T}$  and yet requires an output price movement  $M$  of size  $\Omega(2^{|\mathcal{T}|})$ ?<sup>4</sup> Are there other families of probabilistic trades that can be handled quickly? And finally, can the model be improved? For instance, a single  $+1/-1$  movement might have an influence on a trade. If a price goes up, just below a winning price, then starts going back down, the trader might get scared and close his trade. More realistic trader models are still open.

## Appendix

We prove Lemma 6 here, but we need two intermediate results beforehand.

<sup>4</sup> Note however that a positive answer to this question does not rule out that  $M$  could be represented in compact form (e.g. as code describing  $M$ 's movement) just as the input.

► **Lemma 12.** *Let  $X_1, \dots, X_n \in \mathbb{R}$  with  $n \geq 1$ , and suppose that  $\sum_{i=1}^{n'} iX_i \leq 0$  for every  $1 \leq n' \leq n$ . Then for any reals  $a, c > 0$ ,  $\sum_{i=1}^n (a + ci)iX_i \geq \sum_{i=1}^n (a + cn)iX_i$ .*

**Proof.** We prove the statement by induction over  $n$ . The base case  $n = 1$  is trivial. For larger  $n$ , we suppose by induction that  $\sum_{i=1}^{n-1} (a + ci)iX_i \geq \sum_{i=1}^{n-1} (a + c(n-1))iX_i = \sum_{i=1}^{n-1} ((a + cn)iX_i - ciX_i) = \sum_{i=1}^{n-1} (a + cn)iX_i - c \sum_{i=1}^{n-1} iX_i$ .

Adding  $(a + cn)nX_n$  on both ends of the inequality, we get  $\sum_{i=1}^n (a + ci)iX_i \geq \sum_{i=1}^n (a + cn)iX_i - c \sum_{i=1}^{n-1} iX_i$ , which proves the statement as we assume  $\sum_{i=1}^{n-1} iX_i \leq 0$  and  $c > 0$ . ◀

► **Lemma 13.** *Let  $X_1, \dots, X_n \in \mathbb{R}$  with  $n \geq 1$ , and suppose that  $\sum_{i=1}^n iX_i > 0$ . Then for any reals  $a, c > 0$ , there exists  $m \leq n$  such that  $\sum_{i=1}^m (a + ci)iX_i > 0$ .*

**Proof.** We use induction again. The case  $n = 1$  is trivial. Assume now that the statement holds for values smaller than  $n$ . If there is a  $n' < n$  such that  $\sum_{i=1}^{n'} iX_i > 0$ , then we can apply induction and find  $m \leq n'$  that satisfies  $\sum_{i=1}^m (a + ci)iX_i > 0$ . If no such  $n'$  exists, then for every  $n'$  between 1 and  $n - 1$ ,  $\sum_{i=1}^{n'} iX_i \leq 0$ . Note that then,  $X_1, \dots, X_{n-1}$  satisfy the condition of Lemma 12. Since  $\sum_{i=1}^{n-1} iX_i + nX_n > 0$ , we must have  $(a + cn)nX_n > -(a + cn) \sum_{i=1}^{n-1} iX_i$ . Suppose now that the statement fails on  $m = n$ , i.e.  $\sum_{i=1}^n (a + ci)iX_i + (a + cn)nX_n \leq 0$ . Then  $(a + cn)nX_n \leq -\sum_{i=1}^{n-1} (a + ci)iX_i$ . Our two complementary bounds on  $(a + cn)nX_n$  imply  $\sum_{i=1}^{n-1} (a + ci)iX_i < (a + cn) \sum_{i=1}^{n-1} iX_i$ . This however contradicts Lemma 12. ◀

**Proof of Lemma 6:** Call a direction change (i.e.  $+1$  followed by  $-1$  or vice-versa) *bad* if it occurs outside of  $\hat{\mathcal{S}}$ . Suppose that all optimal price movements have a bad direction change. Choose such an  $M$  such that (1)  $M$  is minimal (i.e.  $M$  does not change direction without closing trades); (2)  $M$  has a bad direction change at some price  $p \notin \hat{\mathcal{S}}$  the latest (i.e. if  $M$  has a bad direction change at step  $i$ , all optimal price movements have a bad direction change at step at most  $i$ ). Let  $r \in \hat{\mathcal{S}}$  such that  $r$  has the same sign as  $p$ ,  $|r| > |p|$  and  $r$  is the closest possible to  $p$ . Assume without loss of generality that  $r > p > 0$  (the case  $r < p < 0$  is symmetric). Thus  $M$  moves from  $p - 1$  to  $p$  at the  $i$ -th step, then goes to  $p - 1$ , and  $p \notin \hat{\mathcal{S}}$ . The idea is that the (expected) gain from  $p$  to  $p + 1$  is at least the gain made from  $p - 1$  to  $p$ . If this gain is profitable, then moving from  $p$  to  $p + 1$  improves  $M$ . Otherwise, the move from  $p - 1$  to  $p$  was not optimal in the first place, and we can do better than  $M$ . We now make this idea formal.

We assume that we have transformed the problem into an MTL instance, and are optimizing  $\mathcal{T}^*$ . Let  $M^+$  be the price movement obtained by inserting  $+1$  and then  $-1$  after the  $i$ -th step in  $M$ . That is,  $M^+$  moves to  $p$  at step  $i$ , then to  $p + 1$ , then to  $p$ , and then does exactly as  $M$  did afterwards. By our choice of  $M$ ,  $M^+$  is suboptimal (for if it is optimal, either it has no bad direction change or it occurs later than  $M$ ). Similarly, let  $M^-$  be the price movement obtained by deleting the  $i$ -th step from  $M$ . So  $M^-$  reaches price  $p - 1$  at step  $i - 1$ , and then does the same actions as  $M$  from step  $i + 1$  at price  $p - 1$ . It is not hard to see that  $M^+$  and  $M^-$  still close every trade as needed.

For some price  $p'$ , let  $\mathcal{T}^*(p') = \{T \in \mathcal{T}^* : w_T = p' \text{ or } \ell_T = p'\}$ . Because  $p \notin \hat{\mathcal{S}}$ , to each trade  $T \in \mathcal{T}^*(p)$  corresponds a distinct trade  $T^+ \in \mathcal{T}^*(p + 1)$  such that  $T$  and  $T^+$  have the same parameters except that  $T^+$  closes at one unit price higher (i.e. if  $T$  is a buy, then  $T^+ = (o_T, w_T + 1, \ell_T, s_{T+})$  with  $w_T = p$ , and if  $T$  is a sell, then  $T^+ = (o_T, w_T, \ell_T + 1, s_T)$  with  $\ell_T = p$ , with  $s_{T+} = s_T$  by the uniformity of  $\mathcal{T}$ ). Moreover, as  $|\mathcal{T}^*(p)| = |\mathcal{T}^*(p + 1)|$  this correspondence is one-to-one and onto.

For two price movements  $M_1, M_2$ , denote by  $W(M_1, M_2)$  the set of trades of  $\mathcal{T}^*$  won by  $M_1$  and lost by  $M_2$ . Now, for any price pair  $p_1, p_2 \neq p+1$ , the relative order of  $p_1$  and  $p_2$  in  $M$  is preserved by  $M^+$ , i.e.  $p_1$  is reached before  $p_2$  by  $M$  if and only if  $p_1$  is reached before  $p_2$  by  $M^+$ . Thus any trade in  $W(M, M^+)$  or  $W(M^+, M)$  must belong to  $\mathcal{T}^*(p+1)$ . It is then not hard to see that  $T \in W(M, M^+)$  implies that  $T$  is a sell, and  $T \in W(M^+, M)$  implies that  $T$  is a buy. In a similar fashion, any trade in  $W(M^-, M) \cup W(M, M^-)$  is in  $\mathcal{T}^*(p)$ . We can deduce the following claim, for which we omit the details:

► **Claim 13.1.**  $T \in W(M^-, M) \Leftrightarrow T^+ \in W(M, M^+)$  and  $T \in W(M, M^-) \Leftrightarrow T^+ \in W(M^+, M)$ .

Denote  $\Delta(M_1, M_2) = pr_{\mathcal{T}^*}(M_1) - pr_{\mathcal{T}^*}(M_2)$  for two price movements  $M_1$  and  $M_2$ . By our assumptions,  $\Delta(M^+, M) < 0$  and  $\Delta(M, M^-) \geq 0$ . This yields the following:

► **Claim 13.2.**  $\sum_{T \in W(M^-, M)} s_T - \sum_{T \in W(M, M^-)} s_T > 0$

**Proof.** Note that  $\Delta(M^+, M)$  is solely due to the trades in  $W(M, M^+) \cup W(M^+, M)$ , i.e.  

$$\begin{aligned} \Delta(M^+, M) &= \sum_{T^+ \in W(M^+, M)} (pr_{T^+}(M^+) - pr_{T^+}(M)) + \sum_{T^+ \in W(M, M^+)} (pr_{T^+}(M^+) - pr_{T^+}(M)) \\ &= \sum_{T^+ \in W(M^+, M)} (p+1 - \ell_{T^+})s_{T^+} + \sum_{T^+ \in W(M, M^+)} (-(p+1) + w_{T^+})s_{T^+} < 0. \end{aligned}$$

In a similar manner, the difference between  $M$  and  $M^-$  is

$$\begin{aligned} 0 \leq \Delta(M, M^-) &= \sum_{T \in W(M, M^-)} (p - \ell_T)s_T + \sum_{T \in W(M^-, M)} (-p + w_T)s_T \\ &= \sum_{T^+ \in W(M^+, M)} (p - \ell_{T^+})s_{T^+} + \sum_{T^+ \in W(M, M^+)} (-p + w_{T^+})s_{T^+} \\ &= \Delta(M^+, M) - \sum_{T^+ \in W(M^+, M)} s_{T^+} + \sum_{T^+ \in W(M, M^+)} s_{T^+} \\ &< -\sum_{T^+ \in W(M^+, M)} s_{T^+} + \sum_{T^+ \in W(M, M^+)} s_{T^+} \end{aligned}$$

where the second equality is due to Claim 13.1 and the uniformity of  $\mathcal{T}$ .

We get  $\sum_{T^+ \in W(M, M^+)} s_{T^+} - \sum_{T^+ \in W(M^+, M)} s_{T^+} > 0$  and Claim 13.2 follows by applying Claim 13.1 on this expression. ◀

In words, what is won by  $M^-$  has a greater total weight than what is won by  $M$ . We will use that to our advantage and derive a better price movement. Let  $z \leq 0$  be the minimum price reached by  $M$  (and  $M^-$ ) before step  $i$ , and let  $q < z$  be the minimum price reached by  $M^-$  after the  $i$ -th step but before reaching price  $p$ . Thus going from  $p-1$  to  $q$  in  $M^-$  is responsible for all the trades in  $W(M^-, M) \cup W(M, M^-)$ .

Consider  $q'$  such that  $q \leq q' < z$ , and let  $M^{q'}$  be the price movement that mimics  $M^-$  until step  $i-1$ , reaching  $p-1$ , then goes down to  $q'$ , then to  $p$ , and finally does exactly what  $M$  does from step  $i$ . Observe that in particular,  $M^q = M$ . It is not hard to check that  $M^{q'}$  closes every trade as required.

We will count  $\Delta(M^{q'}, M)$ , but a bit differently than above. Let  $T_U \in \mathcal{T}$  be one of the input uniform trades. As  $\mathcal{T}$  is uniform, all deterministic trades  $T \in T_U^*$  have the same size  $\frac{1}{|\hat{w}_{T_U}| |\hat{\ell}_{T_U}|} s_{T_U}$ . Denote this size by  $s_{T_U}^*$ . Suppose now that  $T_U$  is a buy trade with  $\hat{w}_{T_U} = p$  and  $\hat{\ell}_{T_U} < z$ . Then for every price  $q''$  with  $\max(\hat{\ell}_{T_U}, q') \leq q'' < z$ , the deterministic trade  $T_U^*(p, q'')$  is in  $W(M, M^{q'})$ . For  $q' < k < z$ , let  $L_k$  be the set of trades  $T_U$  of  $\mathcal{T}$  such that  $\hat{\ell}_{T_U} = k$ , and let  $L_{q'}$  be those trades  $T_U$  of  $\mathcal{T}$  with  $\hat{\ell}_{T_U} \leq q'$ . One can verify that  $W(M, M^{q'}) = \bigcup_{i=1}^{z-q'} \bigcup_{T_U \in L_{z-i}} \bigcup_{j=0}^{i-1} T_U^*(p, z-i+j)$ . In particular, setting  $q' = q$  and using the fact that  $W(M, M^q) = W(M, M^-)$ , we can deduce  $\sum_{T \in W(M, M^-)} s_T = \sum_{i=1}^{z-q} \sum_{T_U \in L_{z-i}} i s_{T_U}^*$ .

As for  $W(M^{q'}, M)$ , let  $W_k$  be the trades of  $T_U$  in  $\mathcal{T}$  such that  $\hat{w}_{T_U} = k$ , and  $W_{q'}$  those with  $\hat{w}_{T_U} \leq q'$ . Similarly as above, we have

$$W(M^{q'}, M) = \bigcup_{i=1}^{z-q'} \bigcup_{T_U \in W_{z-i}} \bigcup_{j=0}^{i-1} T_U^*(p, z-i+j).$$

Since  $W(M^q, M) = W(M^-, M)$ , we have  $\sum_{T \in W(M^-, M)} s_T = \sum_{i=1}^{z-q} \sum_{T_U \in W_{z-i}} i s_{T_U}^*$ . Importantly, Claim 13.2 now implies  $\sum_{i=1}^{z-q} \sum_{T_U \in W_{z-i}} i s_{T_U}^* - \sum_{i=1}^{z-q} \sum_{T_U \in L_{z-i}} i s_{T_U}^* > 0$  (\*).

We now calculate  $\Delta(M^{q'}, M)$  by considering  $W(M^{q'}, M)$  and  $W(M, M^{q'})$  separately.

$$\begin{aligned} \sum_{T \in W(M^{q'}, M)} pr_T(M^{q'}) - pr_T(M) &= \sum_{i=1}^{z-q'} \sum_{T_U \in W_{z-i}} \sum_{j=0}^{i-1} (pr_{T_U^*(p, z-i+j)}(M^{q'}) - pr_{T_U^*(p, z-i+j)}(M)) \\ &= \sum_{i=1}^{z-q'} \sum_{T_U \in W_{z-i}} \sum_{j=0}^{i-1} (-(z-i+j) + p) s_{T_U}^* \\ &= \sum_{i=1}^{z-q'} \sum_{T_U \in W_{z-i}} i s_{T_U}^* (p - z + 1/2 + i/2) \\ &= \sum_{i=1}^{z-q'} (p - z + 1/2 + i/2) \sum_{T_U \in W_{z-i}} i s_{T_U}^* \end{aligned}$$

and for  $W(M, M^{q'})$ ,

$$\begin{aligned} \sum_{T \in W(M, M^{q'})} pr_T(M^{q'}) - pr_T(M) &= \sum_{i=1}^{z-q'} \sum_{T_U \in L_{z-i}} \sum_{j=0}^{i-1} (z-i+j-p) s_{T_U}^* \\ &= \sum_{i=1}^{z-q'} (z-p-1/2-i/2) \sum_{T_U \in L_{z-i}} i s_{T_U}^* \end{aligned}$$

Letting  $K = p - z + 1/2$ , adding these two values, we get

$$\begin{aligned} \Delta(M^{q'}, M) &= \sum_{i=1}^{z-q'} (K + i/2) \sum_{T_U \in W_{z-i}} i s_{T_U}^* + \sum_{i=1}^{z-q'} (-K - i/2) \sum_{T_U \in L_{z-i}} i s_{T_U}^* \\ &= \sum_{i=1}^{z-q'} (K + i/2) \left[ \sum_{T_U \in W_{z-i}} i s_{T_U}^* - \sum_{T_U \in L_{z-i}} i s_{T_U}^* \right] \end{aligned}$$

We are almost done here. Because if we let  $n = z - q$  and  $X_i = \sum_{T_U \in W_{z-i}} i s_{T_U}^* - \sum_{T_U \in L_{z-i}} i s_{T_U}^*$ , by (\*) we have  $\sum_{i=1}^n i X_i > 0$ , and by Lemma 13 there exists a  $n' = q' - z$  such that  $\Delta(M^{q'}, M) = \sum_{i=1}^{n'} (K + i/2) i X_i > 0$ , contradicting the optimality of  $M$ . ◀

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